An introduction to number theory and Diophantine equations: The definition of genera

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Given a prime q, for every m such that $q \nmid m$, define

$$\chi_q(m) = \left(\frac{m}{q}\right).$$

Then χ_q is what is known as a character modulo q. We will use these characters for forms with discriminant D, where $q|D$.

If $2|D$ we must consider additional characters. Let $\langle a, b, c \rangle$ be a quadratic form of discriminant D, and let m be any odd integer representable by $\langle a, b, c \rangle$. Note that if $2|D$ then $2|b^2$ so that in fact $4|b^2$, and hence $4|D$. Thus we define several cases: if $D/4 \equiv 0, 3 \pmod 4$, let

$$\delta(m) = (-1)^{(m-1)/2}.$$

If $D/4 \equiv 0, 2 \pmod 8$, let

$$\epsilon(m) = (-1)^{(m^2-1)/8}.$$

If $D/4 \equiv 0, 6 \pmod 8$, let

$$\delta(m)\epsilon(m) = (-1)^{\frac{m-1}{2} + \frac{m^2-1}{8}}.$$

Then δ is a character modulo 4, ϵ is a character modulo 8, and $\delta\epsilon$ is a character modulo 8.

Let D be a discriminant. We can write $D = S^2 D_0$ where D_0 is square-free, and we call D_0 the square-free kernel of D. Let $q_1, \ldots, q_r$ be the distinct odd primes dividing the square-free kernel of D. Order these primes in such a manner that

$$q_1 \equiv \cdots \equiv q_s \equiv 1 \pmod 4$$

and

$$q_{s+1} \equiv \cdots \equiv q_r \equiv 3 \pmod 4.$$

For each of these primes, define the character $\chi_{q_j}(m)$, which we will call $\chi_j(m)$ for short.

To each type of discriminant D we assign a set of characters that we will use to compute the character system of quadratic forms of discriminant D. We summarize the characters we assign in the following table.

Discriminant	Assigned characters	(t, ω)
$D \equiv 1 \pmod 4$	$\chi_1, \ldots, \chi_r$	(r, r)
$D = 4D_0, D_0 \equiv 1 \pmod 4$	$\chi_1, \ldots, \chi_r$	$(r, r+1)$
$D = 4D_0, D_0 \equiv 3 \pmod 4$	$\chi_1, \ldots, \chi_r, \delta$	$(r+1, r+1)$
$D = 4D_0, D_0 \equiv 2 \pmod 8$	$\chi_1, \ldots, \chi_r, \epsilon$	$(r+1, r+1)$
$D = 4D_0, D_0 \equiv 6 \pmod 8$	$\chi_1, \ldots, \chi_r, \delta\epsilon$	$(r+1, r+1)$
$D = 4D_0, D_0 = 4S^2 q_1 \cdots q_r$	$\chi_1, \ldots, \chi_s, \chi_{s+1}\delta, \ldots, \chi_r\delta$	$(r, r+1)$
$D = 4D_0, D_0 = 8S^2 q_1 \cdots q_r$	$\chi_1, \ldots, \chi_s, \chi_{s+1}\delta, \ldots, \chi_r\delta, \epsilon$	$(r+1, r+1)$

Here the column titled (t, ω) uses the convention that the first entry t denotes the number of assigned characters, while ω denotes the number of distinct primes dividing D.

Note: this formulation is taken from the book *My Numbers, My Friends* by Paulo Ribenboim.